TRINITY COLLEGE

Mathematics Methods Unit 3/4
Test 5 2022

Section 1 Calculator Free
CRVs and Normal Distribution

STUDENT'S NAME SOLUTIONS

DATE: Tuesday 9th August **TIME:** 15 minutes **MARKS**: 14

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser, approved Formula sheet

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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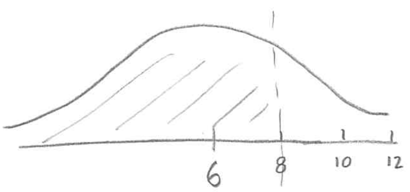
1. (3 marks)

Let X be a normally distributed random variable with a mean of 6 and a variance of 4. Let Z be a random variable with the standard normal distribution.

$\sigma = 2$

(a) Determine $P(X < 8)$. $= 0.5 + 0.34$ ✓ [2]

$= 0.84$

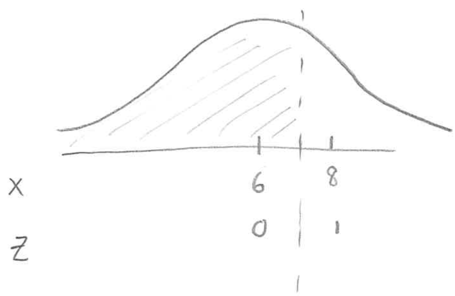


✓ correct σ

✓ 0.84

(b) Determine b such that $P(X < 7) = P(Z < b)$. [1]

$b = 1/2$



✓ $b = 1/2$

2. (5 marks)

The probability density function for a continuous random variable X is given by

$$f(x) = \begin{cases} ax(4-x) & 0 \le x \le 4 \\ 0 & elsewhere \end{cases}$$

(a) Show that the value of a is $\frac{3}{32}$.

[3]

$$\int_0^4 4ax - ax^2 = 1$$

$$\left[2ax^2 - \frac{ax^3}{3}\right]_0^4 = 1$$

$$32a - \frac{64a}{3} = 1$$

$$96a - 64a = 3$$

$$32a = 3$$

$$a = 3/32$$

✓ correct integral statement

✓ integrating correctly

✓ working for 'a'

(b) Determine $P(X < 3)$

[2]

$$\frac{3}{32}\int_0^3 4x - x^2 \quad = \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \frac{3}{32}(18 - 9 - 0)$$

$$= \frac{27}{32}$$

✓ correct integral

✓ $\frac{27}{32}$

3. (6 marks)

The continuous random variable X takes values in the interval 1 to 5 and has cumulative distribution function $F(x)$ where

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 1 \\ \dfrac{x-1}{4} & 1 \le x \le 5 \\ 1 & x > 5. \end{cases}$$

(a) Determine

(i) $P(X \le 3.5)$ [1]

$= \dfrac{3.5 - 1}{4}$

$= \dfrac{5}{8}$ (0.625)

✓ correct answer

(ii) the value of k, if $P(X > k) = 0.85$ [2]

$P(X < k) = 0.15$

$\dfrac{k-1}{4} = 0.15$

$k - 1 = 0.6$

$k = 1.6$

✓ correct algebraic statement

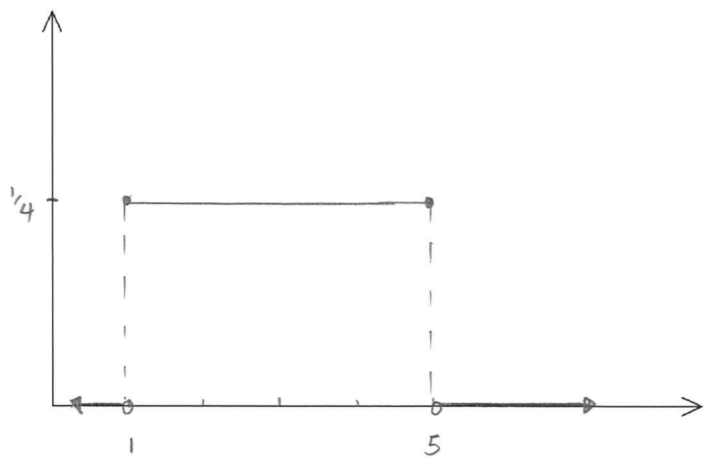
✓ $k = 1.6$

(b) Determine $f(x)$, the probability density function of X, and sketch the graph of $y = f(x)$. [3]

$$f(x) = \begin{cases} 0 & x < 1 \\ 1/4 & 1 \le x \le 5 \\ 0 & x > 5 \end{cases}$$

✓ $1/4$ $1 \le x \le 5$

✓ 0 elsewhere



✓ sketch (must include values on axis)

TRINITY COLLEGE

Mathematics Methods Unit 3/4
Test 4 2022

Section 2 Calculator Assumed
CRVs and Normal Distribution

STUDENT'S NAME ______________________________

DATE: Tuesday 9th August **TIME:** 25 minutes **MARKS**: 27

INSTRUCTIONS:

Standard Items:	Pens, pencils, drawing templates, eraser, approved Formula sheet
Special Items:	Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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4. (4 marks)

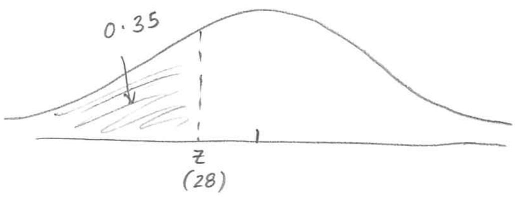
The continuous random variable, X, is normally distributed with $P\,(X < 28) = 0.35$

(a) How many standard deviations from the mean is a score of 28? [2]

$z = -0.385$

✓ z value

✓ correct description



0·385 s.d to the left of the mean (below)

(b) If the standard deviation of X is 5.74 find the mean of the distribution, giving your answer correct to 2 decimal places. [2]

$$-0{\cdot}385 = \frac{28 - \mu}{5{\cdot}74}$$

✓ using z rule

$\mu = 30{\cdot}21$

✓ $\mu = 30{\cdot}21$ (2dp)

5. (4 marks)

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} \dfrac{x}{12} & 1 \leq x \leq 5 \\ 0 & elsewhere \end{cases}$$

(a) Explain why f is a probability density function. [1]

$$\int_1^5 x/12 \, dx = 1$$

✓ statement (or words)

(b) Determine $P(X < 3)$. [1]

$$\int_1^3 \frac{x}{12} \, dx = \frac{1}{3}$$

✓ $\frac{1}{3}$

(c) If $P(X \geq a) = \frac{5}{8}$, find the value of a. [2]

$$\int_a^5 \frac{x}{12} \, dx = \frac{5}{8}$$

✓ correct integral

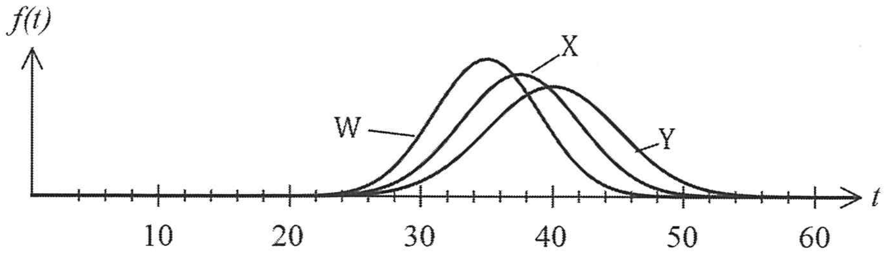
$a = \sqrt{10}$

$(a = 3.16)$

✓ a = correct value

6. (8 marks)

The graphs of the probability density functions of three normally distributed random variables W, X and Y are shown below.



(a) State, with justification, which of the three random variables has

(i) the largest standard deviation? [1]

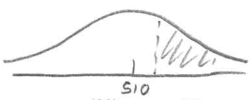
Y most spread ✓ Y and reason

(ii) the largest mean? [1]

Y largest middle score ✓ Y and reason

(b) Empty bottles are filled with A mL of water, where A is a normally distributed random variable with mean of 510 mL and standard deviation of 7.5 mL.

(i) Determine the probability that a bottle is filled with more than 520 mL. [1]



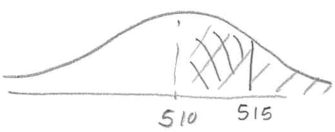
$P(A > 520) = 0.0912$ ✓ 0.0912

(ii) Determine the probability that a bottle is filled with less than 515 mL, given that it is filled with more than 510 mL. [2]

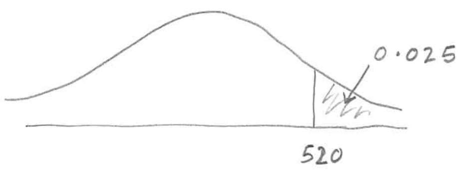
$$\frac{P(510 < A < 515)}{P(A > 510)} = \frac{0.2475}{0.5}$$

$= 0.495$

✓condition
✓0.9900



(iii) The mean of A is to be decreased by k mL so that just 2.5% of bottles are filled with 520 mL or more. Determine the value of k. [3]



$z = 1.96$

$$1.96 = \frac{520 - \mu}{7.5}$$

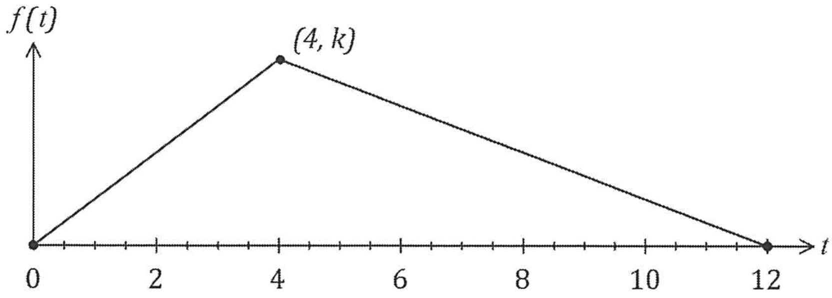
$\mu = 505.3$

$\therefore k = 4.7$ mL

✓ z value
✓ μ
✓ k

7. (11 marks)

The time T to process orders at a warehouse is a random variable which can take any value in the interval 0 to 12 minutes. The graph of the triangular probability density function of T is shown below.



(a) Determine the value of k. [1]

$1 = \frac{1}{2}(12)(k)$

$k = 1/6$

✓ k

(b) Determine the probability that the time to process an order takes less than 3 minutes. [3]

$m = \frac{\frac{1}{6}}{4} = \frac{1}{24}$

$\int_0^3 \frac{1}{24} t \, dt = 0.1875$

✓ eqn of line

✓ integral

✓ 0.1875

(c) Determine the mean time to process an order in minutes and seconds. [4]

$$\int_0^4 t\left(\frac{1}{24}t\right)dt + \int_4^{12} -\frac{t}{48}(t-12)\,dt$$

$$= \frac{8}{9} + \frac{40}{9}$$

$$= \frac{16}{3} \quad \left(5\tfrac{1}{3},\ 5.\dot{3}\right)$$

$= 5$ mins 20 secs

$$m = \frac{\frac{1}{6}}{8} = \frac{1}{48}$$

$$y = -\frac{1}{48}x + c$$

$$0 = -\frac{12}{48} + c$$

$$c = \frac{12}{48}$$

$$y = -\frac{1}{48}(x-12)$$

✓ eqn of line

✓ both integrals

✓ value

✓ value → time (min, secs)

The variance of T is 6 minutes 13 seconds.

(d) Two new procedures will affect the processing time of an order. The first will decrease the time by 15% and the second will then add one-and-a-half minutes. Determine the new mean and standard deviation of the time to process an order. [3]

$$E(T) = 0.85\left(5\tfrac{1}{3}\right) + 1.5$$

$$= 6.0\dot{3}$$

$= (6$ min 2 secs$)$

✓ new mean

✓ old s.d

✓ new s.d

old $\sigma = \sqrt{6\,^{13}/_{60}} = 2.49$

new $\sigma = 2.49 \times 0.85$

$= 2.12$

(2 mins 7 secs)